# MATH 4377 - MATH 6308

**Demetrio Labate**
dlabate@uh.edu

# Outline

1. Chapter 2
   - Section 2.3 - Composition of Linear Transformations
   - Section 2.4 - Invertibility and Isomorphism
   - Section 2.5 - Change of Coordinate Matrix
   - Section 2.6 - Dual Spaces

# Composition of Linear Transformations and Matrix Multiplication

Section 2.3

# Theorem about the composition of linear transformations

**Theorem**

Let $V, W, Z$ be vector spaces over the same field. Let $T : V \to W$ and $U : W \to Z$ be linear. Then $U \circ T : V \to Z$ is linear.

# Theorem about the composition of linear transformations

**Theorem**

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*Proof.*

We want to verify that linearity holds for $U \circ T$:

(i) $U \circ T(\alpha v) = U(T(\alpha v)) = U(\alpha T(v)) = \alpha U(T(v))$

(ii)
$U \circ T(v + v') = U(T(v + v')) = U(T(v) + T(v')) = U(T(v)) + U(T(v'))$

# Properties of the composition of linear transformations

**Theorem**

Let $U, S, T : V \to V$ linear. Then

- $U \circ (S + T) = U \circ S + U \circ T$
- $(U + S) \circ T = U \circ T + S \circ T$
- $U \circ (S \circ T) = (U \circ S) \circ T$
- $I \circ U = U \circ I = U \quad (I$ is the identity)
- $a(U \circ S) = (aU) \circ S = U \circ (aS) \quad (a$ scalar)

# Matrix of a composition

**Goal:** Want to write the matrix representation of $U \circ T$.

**Theorem**

Let $T : V \to W$, $U : W \to Z$ be linear transformations on finite-dimensional vector spaces. Let $\alpha = \{\mathbf{v}_j\}, \beta = \{\mathbf{w}_k\}, \gamma = \{\mathbf{z}_i\}$ be the ordered basis for $V$, $W$, and $Z$. Then $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

# Matrix of a composition

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*Proof.*

We can write the matrices $[T]_\alpha^\beta = (b_{kj})$ and $[U]_\beta^\gamma = (a_{ik})$. Then

$$(U \circ T)(\mathbf{v}_j) = U(T(\mathbf{v}_j)) = U\left(\sum_k b_{kj}\mathbf{w}_k\right) = \sum_k b_{kj} U(\mathbf{w}_k)$$

$$= \sum_k b_{kj}\left(\sum_i a_{ik}\mathbf{z}_i\right) = \sum_i \left(\sum_k a_{ik} b_{kj}\right)\mathbf{z}_i.$$

Consequently, the matrix representation of $[U \circ T]_\alpha^\gamma$ is $c_{ij} = \sum_k a_{ik} b_{kj}$.

# Matrix product

## Definition

Let $A$ be an $m \times n$ matrix and $B$ an $n \times p$ matrix. Define the *matrix product* of $A$ and $B$ to be the $m \times p$ matrix $C = (c_{ij})$ given by

$$(AB)_{ij} = c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

**Ex:**

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} =$$

# Matrix notation

Let $A = (a_{ij})$ be an $m \times n$ matrix.

We can denote the matrix explicitly as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Sometimes it is useful to represent the matrix in terms of its columns

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{pmatrix}$$

Note: the matrix product is not commutative!

## Definition

Let $A \in \mathrm{M}_{m\times n},\ B, C \in \mathrm{M}_{n\times p},\ D, E \in \mathrm{M}_{q\times m}$. Then

1. $A(B + C) = AB + AC$
2. $(D + E)A = DA + EA$
3. $a(AB) = A(aB) = (aA)B \quad (a \text{ scalar})$
4. If $V$ is an $n$ dimensional vector space with ordered basis $\beta$, then $[I_V]_\beta = I_n$.
5. $I_m A = A I_n$

**Theorem**

Let $V, W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$ and $T : V \to W$ be linear. Then $\forall \mathbf{v} \in V$ :

$$[T(\mathbf{v})]_\gamma = [T]_\beta^\gamma [\mathbf{v}]_\beta.$$

## Theorem

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*Proof.* Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$.
We can expand any $v \in V$ w.r. to $\beta$, so it will be sufficient to examine one basis vector. For $J$ fixed, we can write

$$T(v_J) = \sum_{i=1}^{m} a_{iJ} w_i$$

Hence

$$[T(v_J)]_\gamma = (a_{1J}, \ldots, a_{mJ})$$

and

$$[T]_\beta^\gamma [v_J]_\beta = (a_{ij}) e_J = (a_{1J}, \ldots, a_{mJ}),$$

where $(a_{ij})e_J$ is extracting the $J$-th column of the matrix.

# Matrix vector multiplication

Multiplying a $m \times n$ matrix $A$ and a vector $\mathbf{x} \in F^n$ transforms $x$ into a vector $A\mathbf{x} \in F^n$.

Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

showing that $A\mathbf{x}$ **is a linear combination of the columns of** $A$**.**

# Matrix vector multiplication

Let $A$ be a $m \times n$ matrix, $B$ a $n \times p$ matrix and $\mathbf{x} \in F^p$ a vector.
We use the notation $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{pmatrix}$

We have that $ABx = A(Bx)$, so that $A$ acts on the $n$-vector $Bx$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$AB\mathbf{x} = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \cdots + x_p A\mathbf{b}_p = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Hence $AB = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{pmatrix}$, showing that:
**each column of $AB$ is a linear combination of the columns of $A$ using weights from the corresponding columns of $B$.**

# Left-multiplication transformation

## Definition

Let $A \in \mathrm{M}_{m\times n}(F)$. Define $L_A : F^n \to F^m$ by

$$L_A(\mathbf{x}) = A\mathbf{x},$$

where $\mathbf{x} \in F_n$ is a column vector.
$L_A$ is the **left-multiplication transformation** given by the matrix A.

# Properties of left-multiplication transformation

**Theorem**

Let $A, B$ be $m \times n$ matrix and $\beta, \gamma$ be the standard ordered bases of $F^n$ and $F^m$, then:

1. $L_A : F^n \to F^m$ is linear.
2. $[L_A]_\beta^\gamma = A$
3. $L_A = L_B \Leftrightarrow A = B$
4. $L_{A+B} = L_A + L_B$,
5. $L_{aA} = aL_A \quad a \in F$
6. For $T : F^n \to F^m$, there exists a unique $m \times n$ matrix $C$ such that $T = L_C$ and $[T]_\beta^\gamma = C$.
7. If $m = n$, then $L_{I_n} = I_{F_n}$

# Invertibility and Isomorphism

Section 2.4

# Invertible transformation

## Definition

Let $V, W$ be vector spaces. Let $T : V \to W$ linear. We define $U : W \to V$ to be the *inverse* of $T$ if $T \circ U = I_W$ and $U \circ T = I_V$.
If $T$ has an inverse, $T$ is called *invertible*.

# Invertible transformation

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**Remarks about continuous functions from Appendix B:**

- If $f$ has an inverse, the inverse is unique. We write $f^{-1}$ for it.
- Given sets $A, B$, a function $f : A \to B$ is invertible if and only if $f$ is one-to-one and onto.

This observation applies to linear transformations.

# Example

Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be defined as $T(a + bx) = (a, a + b)$. Verify that:

$$T^{-1} : \mathbb{R}^2 \to P_1(\mathbb{R}), \;\; T^{-1}(c, d) = c + (d - c)x$$

# Example

Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be defined as $T(a+bx) = (a, a+b)$. Verify that:

$$T^{-1} : \mathbb{R}^2 \to P_1(\mathbb{R}),\ \ T^{-1}(c,d) = c + (d-c)x$$

SOLUTION.
(i) $(T^{-1} \circ T)(a+bx) = T^{-1}(T(a+bx)) = T^{-1}(a, a+b) =$
$a + (a+b-a)x = a + bx$

(ii)
$(T \circ T^{-1})(c,d) = T(T^{-1}(c,d)) = T(c+(d-c)x) = (c, c+d-c) = (c,d)$

# Theorem

**Theorem**

If $T : V \to W$ is linear and invertible, then the inverse $T^{-1}$ is linear also.

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*Proof.*
(i) For $v \in V$, let $T(v) = w$. For $c \in F$,

$$T^{-1}(cw) = T^{-1}(cT(v)) = T^{-1}(T(cv)) = cv = cT^{-1}(w).$$

(ii) For $v_1, v_2 \in V$, let $T(v_1) = w_1$, $T(v_2) = w_2$.

$$\begin{aligned} T^{-1}(w_1 + w_2) &= T^{-1}(T(v_1) + T(v_2)) = T^{-1}(T(v_1 + v_2)) = v_1 + v_2 \\ &= T^{-1}(w_1) + T^{-1}(w_2). \end{aligned}$$

# Lemma

## Lemma

Let $T : V \to W$ linear and invertible. Let $V, W$ be finite dimensional. Then

$$\dim V = \dim W.$$

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*Proof.*
*Since $T$ is invertible, it is one-to-one and onto, hence nullity$(T) = 0$ and rank$(T) = \dim W$.*
*By the dimension theorem,*

$$nullity(T) + rank(T) = \dim V.$$

*This implies that* $\dim W = rank(T) = \dim V$.

# Invertible matrix

## Definition

Let $A$ be an $n \times n$ matrix. Say $A$ is *invertible* if there exists an $n \times n$ matrix $B$ such that

$$AB = I_n = BA.$$

**Remark:** Such a $B$ is unique, if it exists. Thus, we can write $A^{-1}$ for $B$.

# Recall

Provided $ad - cb \neq 0$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Ex:** Given

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix},$$

find $A^{-1}$.

# Theorem

## Theorem

Let $V, W$ be finite dimensional vector spaces with ordered bases $\beta, \gamma$ respectively. Let $T : V \to W$ linear. Then $T$ is invertible if and only if $[T]_\beta^\gamma$ is an invertible matrix.
Moreover, in this case, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

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Moreover, in this case, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

*Proof.*
The transformation $T : V \to W$ has matrix representation $[T]_\beta^\gamma$
If $T$ is invertible, then it is a linear transformation $T^{-1} : W \to V$ with matrix representation $[T^{-1}]_\gamma^\beta$
This also implies that $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$, so that the matrix $[T]_\beta^\gamma$ is invertible.

Converse direction is left for exercise.

# Corollaries

## Corollary

Let $V$ be a finite dimensional vector spaces with ordered basis $\beta$ respectively. Let $T : V \to V$ linear. Then $T$ is invertible if and only if $[T]_\beta$ is an invertible matrix.
Moreover, in this case, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

## Corollary

An $n \times n$ matrix $A$ is invertible if and only if $L_A$ is invertible and, in this case, $(L_A)^{-1} = L_{A^{-1}}$.

# Isomorphism

**Definition**

Let $V, W$ be vector spaces. Say $V$ is *isomorphic* to $W$ if and only if there exists $T : V \to W$ linear and invertible.
Such a $T$ is called an *isomorphism* from $V$ onto $W$.

**Remark:** The property of being isomorphic is an equivalence relation on the set of vector spaces over a given field.
Informally, a vector space $V$ is isomorphic to $W$ if every vector space calculation in $V$ is accurately reproduced in $W$, and vice versa.

# Theorem

**Theorem**

Let $V, W$ be finite dimensional vector spaces (over the same field). Then $V$ is isomorphic to $W$ if and only if $\dim V = \dim W$.

*Proof for* $\Rightarrow$ This was proved by Lemma in slide 23.

*Proof for* $\Leftarrow$ Suppose $\dim V = \dim W$. We need to show that there exists a linear and invertible map $T : V \to W$.
Let $B = \{v_1, \dots, v_n\}$ be a basis of $V$ and $G = \{w_1, \dots, w_n\}$ a basis of $W$
We define a linear transformation $T : V \to W$ by $T(v_i) = w_i$, for $i = 1, \dots, n$.
Since $span\{T(v_i) : i = 1, \dots, n\} = span\{w_i : i = 1, \dots, n\} = W$, then $T$ is onto.
By the Dimension Theorem, $nullity(T) + \text{rank}(T) = \dim V$ implying that $nullity(T) + \dim W = \dim V$ so that $nullity(T) = 0$. This implies that $T$ is one-to-one.
Since $T$ is one-to-one and onto, it is invertible.

# Corollary

**Corollary**

Let $V$ be a vector space over $F$. Then $V$ is isomorphic to $F^n$ if and only if $\dim V = n$.

Examples of vector space isomorphic to $\mathbb{R}^n$:

1. $P_{n-1}$ the set of polynomial of degree at most $n-1$
2. $M^{p\times q}$, $pq = n$, the class $p \times q$ matrices where $pq = n$.

In these examples one can construct an invertible linear map from $\mathbb{R}^n$ into these spaces.

# Theorem

## Theorem

Let $V, W$ be finite dimensional vector spaces over $F$. Let $\dim V = n$, $\dim W = m$. Let $\beta, \gamma$ be ordered bases for $V$ and $W$. Then

$$\Phi : \mathcal{L}(V, W) \to M_{m \times n}(F)$$

is an isomorphism.

*Proof.* We need to show that $\Phi$ is an isomorphism.
A direct argument shows that $\Phi$ is linear. In fact, for any $T, U \in \mathcal{L}(V, W)$, with matrix representations $[T]_\beta^\gamma$ and $[U]_\beta^\gamma$, respectively, where $\beta = \{v_1, \ldots, v_n\}$ is a basis of $V$ and $\gamma = \{w_1, \ldots, w_m\}$ a basis of $W$, we have that

$$[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$$

$$[cT]_\beta^\gamma = c[T]_\beta^\gamma$$

# (continue)

We also need to show that $\Phi$ is invertible.

The nullspace is $N(\Phi) = \{T : V \to W : [T]_\beta^\gamma = 0^{m \times n}\}$ and consists of the linear transformations mapping every vector to the 0 vector. This space contains only the transformation mapping every vector ot 0, hence $\Phi$ is one-to-one.

For every matrix $A = [a_{ij}]$, $A \in M^{m \times n}$ we can construct the linear transformation

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i, \quad j = 1, \ldots, n$$

showing that $\Phi$ is onto.
Since $\Phi$ is one-to-one and onto, it is invertible.

**Remark:** $\dim \mathcal{L}(V, W) = \dim M_{m \times n} = n \cdot m = (\dim \text{V}) \cdot (\dim W)$.

# The standard representation

## Definition

let $\beta$ be be ordered basis for an $n$-dimensional vector space $V$ over the field $F$. The standard representation of $V$ with respect to $\beta$ is the function

$$\phi_\beta : V \to F^n$$

given by $\phi_\beta(x) = [x]_\beta$.

## Theorem

For any finite dimensional vector space $V$ with ordered basis $\beta$, the map $\phi_\beta$ is an isomorphism.

# The standard representation

**Example.**

Let $\beta = \{b_1, b_2\}$ where $b_1 = (3, 3, 1)$, $b_2 = (0, 1, 3)$.
Let $H = span\{b_1, b_2\}$. Find the standard representation of $H$ with respect to $\beta$ for $x = (9, 13, 15)$.

We need to find $c_1, c_2$ such that $x = c_1 b_1 + c_2 b_2$.

Solving the equation, we find $\phi_\beta(x) = [x]_\beta$.

# The Change of Coordinate Matrix

Section 2.5

# Theorem

## Theorem

Let $\beta, \beta'$ be ordered bases for the finite-dimensional vector space $V$. Let $Q = [I_V]_{\beta'}^{\beta}$. Then

1. $Q$ is invertible
2. $\forall \mathbf{v} \in V : [\mathbf{v}]_\beta = Q[\mathbf{v}]_{\beta'}$.

**Remark:** $Q$ is called the *change of coordinates matrix*. It changes $\beta'$-coordinates to $\beta$-coordinates.

## Theorem

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**Remark:** $Q$ is called the *change of coordinates matrix*. It changes $\beta'$-coordinates to $\beta$-coordinates.

*Proof.* (1) $Q$ is one-to-one since $N(Q) = \{0\}$ and it is one onto since for any $v$ in the $\beta$ basis we can find a $v$ in the $\beta'$ basis.

(2) $[T(\mathbf{v})]_\gamma = [T]_\beta^\gamma [\mathbf{v}]_\beta$

Also $[I_v(\mathbf{v})]_\beta = [I_v]_{\beta'}^{\beta}[\mathbf{v}]_{\beta'}$

Hence $[\mathbf{v}]_\beta = [I_V]_{\beta'}^{\beta}[\mathbf{v}]_{\beta'}$

# Example

Let $\beta' = \{(2,4),(3,1)\} := \{v_1, v_2\}$, $\beta = \{(1,1),(1,-1)\}$. Find the change of coordinates matrix $Q = [I_V]_{\beta'}^{\beta}$.

# Example

Let $\beta' = \{(2,4),(3,1)\} := \{v_1, v_2\}$, $\beta = \{(1,1),(1,-1)\}$. Find the change of coordinates matrix $Q = [I_V]_{\beta'}^{\beta}$.

SOLUTION:

$I(v_1) = (2,4) = a_1(1,1) + a_2(1,-1) = (a_1 + a_2, a_1 - a_2)$
Solving the system, one finds $a_1 = 3, a_2 = -1)$

$I(v_2) = (3,1) = b_1(1,1) + b_2(1,-1) = (b_1 + b_2, b_1 - b_2)$
Solving the system, one finds $b_1 = 2, b_2 = 1)$

Thus, we have that $Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

# Question

Let $T : V \to V$ linear.
What is the relationship between $[T]_\beta$ and $[T]_{\beta'}$?

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Let $T : V \to V$ linear.
What is the relationship between $[T]_\beta$ and $[T]_{\beta'}$?

Recall that if $T : V \to V$ with basis $\beta$ for both domain and codomain then the natrix representation of $T$ is $[T]_\beta^\beta = [T]_\beta$

# Theorem

**Theorem**

Let $T : V \to V$ linear with $V$ finite dimensional vector space. Let $\beta, \beta'$ be ordered bases for $V$. Let $Q = [I_V]_{\beta'}^{\beta}$. Then

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

# Theorem

## Theorem

Let $T : V \to V$ linear with $V$ finite dimensional vector space. Let $\beta, \beta'$ be ordered bases for $V$. Let $Q = [I_V]_{\beta'}^{\beta}$. Then

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

*Idea of Proof*

$$[T]_{\beta'}^{\beta'} = [I_V]_{\beta'}^{\beta}\,[T]_{\beta}^{\beta}\,[I_V]_{\beta}^{\beta'}$$

# Example

Let $\beta = \{(1,1),(1,-1)\}$, $\beta' = \{(2,4),(3,1)\}$. Knowing that

$$[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix},$$

find $[T]_{\beta'}$.

# Example

Let $\beta = \{(1,1),(1,-1)\}$, $\beta' = \{(2,4),(3,1)\}$. Knowing that

$$[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix},$$

find $[T]_{\beta'}$.

SOLUTION:

From prior example, we have that the change of variables matrix is
$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$.
By theorem above, $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.
We compute $Q^{-1} = \frac{1}{3+2}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/5 & -2/5 \\ 1/5 & 3/5 \end{pmatrix}$.

Therefore:

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} 1/5 & -2/5 \\ 1/5 & 3/5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

Solving the matrix multiplication, we find

$$[T]_{\beta'} = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$$

# Similar matrices

## Definitions

Let $A, B \in M_{n\times n}(F)$. We say $B$ is *similar* to A if and only if there exists an invertible $Q \in M_{n\times n}(F)$:

$$B = Q^{-1}AQ.$$

**Remark:** Being similar is an equivalence relation on $M_{n\times n}(F)$.

# Dual Spaces

Section 2.6

# Linear functional

## Definition

Let $V$ be a vector space over the field $F$. A linear transformation $f : V \to F$, where $F$ is considered as a vector space over itself, is called a *linear functional*.

**Examples:**

1. $f : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto x_1$
2. $f : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto 2x_1 - 3x_2$
3. $f : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$

# More examples

4. $\text{tr} : M_{n\times n} \to \mathbb{R}$, $A \mapsto \text{tr}(A)$

5. $\text{eval}_5 : \{g : \mathbb{R} \to \mathbb{R}\} \to \mathbb{R}$, $g \mapsto g(5)$.

6. Let $V$ be a vector space over $F$, $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ ordered basis. Let $f_i : V \to F$ be defined by:
$$f_i(\mathbf{v}) = \text{the } i\text{-th coordinate of } \mathbf{v} \text{ with respect to } \beta.$$
In other words, if $[\mathbf{v}]_\beta = (a_1, \ldots, a_n)$, then $f_i(\mathbf{v}) = a_i$. In particular, $f_i(\mathbf{v}_j) = \delta_{ij}$.

# Example

Let $V = \mathbb{R}^2$. Let $\beta = \{(2,1),(3,1)\} := \{v_1, v_2\}$. Find $f_1, f_2$.

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SOLUTION.
For $v \in \mathbb{R}^2$, $[v]_\beta = (a_1, a_2)$ and $f_1(v) = a_1, f_2(v) = a_2$.

$(x, y) = a_1(2,1) + a_2(3,1) = (2a_1 + 3a_2, a_1 + a_2)$
Solving the linear system, we obtain:

$$a_1 = -x + 3y,\ a_2 = x - 2y$$

Thus:

$$f_1(v) = -x + 3y,\ f_2(v) = x - 2y$$

# Example

Given $f_1(x,y) = -x + 3y$ and $f_2(x,y) = x - 2y$, find $\beta = (v_1, v_2) \subset \mathbb{R}^2$.

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Given $f_1(x,y) = -x + 3y$ and $f_2(x,y) = x - 2y$, find $\beta = (v_1, v_2) \subset \mathbb{R}^2$.

SOLUTION.
We can write any $v = (x,y) \in \mathbb{R}^2$ as

$$v = f_1 v_1 + f_2 v_2$$

If we choose $f_1 = 1, f_2 = 0$, then $v_1 = f_1 v_1$ yielding the system of equations

$$\begin{aligned} -x + 3y &= 1 \\ x - 2y &= 0 \end{aligned}$$

with solution $x = 2, y = 1$, so that we obtain $v_1 = (2,1)$.
Similarly, choosing $f_1 = 0, f_2 = 1$, then $v_2 = f_2 v_2$ yielding the system of equations

$$\begin{aligned} -x + 3y &= 0 \\ x - 2y &= 1 \end{aligned}$$

with solution $x = 3, y = 1$, so that we obtain $v_2 = (3,1)$.

# Dual space

**Definition**

For a vector space $V$ over $F$, let the *dual space* $V^*$ be $\mathcal{L}(V, F)$.

**Remark:** For a finite-dimensional $V$

$$\dim V^* = \dim \mathcal{L}(V, F) = \dim V \cdot \dim F = \dim V$$

This means that $V$ and $V^*$ are isomorphic as vector spaces over $F$.
Also, we can define $V^{**}$ as the dual of $V^*$.

## Theorem

Let $V$ be a vector space and $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ ordered basis. Let $f_i$ be the $i$-th coordinate function with respect to $\beta$, as defined above.
Then $\beta^* = \{f_1, \ldots, f_n\}$ is an ordered basis for $V^*$ and for all $g \in V^*$,

$$g(\mathbf{v}) = \sum_{i=1}^{n} g(\mathbf{v}_i) f_i.$$

## Definition

We call $\beta^*$ the *dual basis* of $\beta$.